Modeling the Mind-Body Nexus in Soft Embodied Agents

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Abstract

There still is a great gap between the adaptivity, flexibility, and dexterity of intelligent robots, whether they are made of rigid or (partially) compliant components and their naturally evolved animal and vegetal counterparts. One reason could be of 'paradigm' nature. On the one hand, most robots (both rigid and soft) are still designed by following a 'Cartesian' approach, with top-down control and a clear mindbody separation. On the other hand, the principles of organization of animals and plants rely on morphological computation (that can be regarded as a partial or total outsourcing of information processing to the agent's dynamics) and emergence of fit behaviors from loosely coupled-sometimes huge and complex-networks of embodied agents. Designing artificial agents on the basis of the principles that we observe in biology is not an easy task. Modeling and control require radically new approaches. In this short paper we briefly recall and explain some seminal mathematical results linking information metrics and dynamics of the agents, already published years ago; we integrate them with some recent methods developed by us and by the community, we do some brief remarks about their relevance and we hint to our current and near future work towards finding concrete and efficient ways for the modeling and control of completely or partially compliant artificial agents and for a better understanding of physical intelligence in nature.

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***Abstract*— There still is a great gap between the adaptivity, flexibility, and dexterity of intelligent robots, whether they are made of rigid or (partially) compliant components and their naturally evolved animal and vegetal counterparts. One reason could be of 'paradigm' nature. On the one hand, most robots (both rigid and soft) are still designed by following a 'Cartesian' approach, with top-down control and a clear mind-body separation. On the other hand, the principles of organization of animals and plants rely on morphological computation (that can be regarded as a partial or total outsourcing of information processing to the agent's dynamics) and emergence of fit behaviors from loosely coupled - sometimes huge and complex - networks of embodied agents. Designing artificial agents on the basis of the principles that we observe in biology is not an easy task. Modeling and control require radically new approaches. In this short paper we briefly recall and explain some seminal mathematical results linking information metrics and dynamics of the agents, already published years ago; we integrate them with some recent methods developed by us and by the community, we do some brief remarks about their relevance and we hint to our current and near future work towards finding concrete and efficient ways for the modeling and control of completely or partially compliant artificial agents and for a better understanding of physical intelligence in nature.**

I. Background

This work is an integration to [1], which builds on the results in [2]. In this short letter we aim to answer to the question 'How the Body Shapes the Way We Think?', [3]. We aim to answer in a way that allows the understanding of why softness a.k.a. compliance helps in many cases in robotics and in natural embodied agents and to start to develop a modeling and control framework which takes inspiration from the principle of organization of living beings in a very deep and fundamental sense. In the following section II we recall some fundamental results obtained and published more than ten years ago and probably due to maybe their originality, conceptual depth and more likely the technical difficulties to use them in practice, had so far had a limited impact. In the next section III we discuss assumptions and overall meaning of the terms of equation 1 presented here, the more important of the four equations, the one on which we focus in this report. In section IV we discuss what we believe have been in the past the most serious obstacle to the practical utilization of those relations, the calculation of W quantities. We conclude with some brief remarks in section V.

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II. Mind–Body Nexus Equations

The main modeling issue is to link the information processing metrics of the 'mind' of the agent to those of the body's system dynamics.

In [1], [4] the following expressions were derived:

$$K(X) \overset{+}{\leq} \ln \frac{\Delta W closed}{\Delta W \overset{max}{\text{open}}} \tag{1}$$

$$\Delta HN + \sum_{i}^{n} \Delta H_i - \Delta I \leq I(X;C) \tag{2}$$

$$K(X) = \Delta HN + \sum_{i}^{n} \Delta H_i - \Delta I \tag{3}$$

$$\Delta HN = \ln \frac{\Delta \Omega closed}{\Delta \Omega \overset{max}{\text{open}}} + \Delta I \tag{4}$$

The equation (1) follows in particular from:

$$H_{shannon} = -\frac{1}{k_b} H_{boltzman} = \frac{1}{k_b} \ln W \tag{5}$$

For details see [1], [4].

III. Discussion of equation (5): Meaning of W

The concept of entropy comes from statistical mechanics, but in particular if we follow an axiomatic probabilistic instead of a frequentistic approach, it can be generalized to a large class of systems including the biological ones, see [5], [6]. Entropy allows to connect a large number of 'micro' variables to a much smaller number of observed variables. In this case we deduct the expected length of a control program of a dynamical system. Equation (5) holds when all Shannon-Khinchin (S-K) four axioms are valid and the underlying dynamics is then estensive. This is true if we look at the possible paths in the phase space of our systems. The S-K 4th axiom is satisfied as the trajectories of the system in the phase space only depend on its dynamical equations.

The Shannon-Khinchin Axioms

The (Shannon) entropy offers a way to quantify the 'information' carried by a stochastic variable with an associate probability distribution, p(x):

$$H(x) = -\sum_{x \in X} p_x(x) \log_2 p_x(x) \tag{6}$$

It can be proven that any metrics compliant with Shannon-Khinchin axioms listed below must have this form.

1) **Continuity** The metric should be such that if you change the values of the probabilities by a very small amount , the metric should only change by only a small amount.
2) **Symmetry** The metric should not change if the results x_i are re-ordered, i.e.:

$$H_n\left(p_x\left(x_1\right), p_x\left(x_2\right), ..., p_x\left(x_n\right)\right) = H_n\left(p_x\left(x_2\right), p_x\left(x_1\right), ..., p_x\left(x_n\right)\right) \tag{7}$$

3) **Maximum** The metric should be maximal if all the outcomes are equally likely (uncertainty is highest when all possible events are equiprobable)

$$H_n\left(p_x\left(x_1\right), ..., p_x\left(x_n\right)\right) \leq H_n\left(\frac{1}{n}, ..., \frac{1}{n}\right) \tag{8}$$

For equiprobable events the entropy should increase with the number of outcomes:

$$\underbrace{H_n\left(\frac{1}{n}, ..., \frac{1}{n}\right)}_{n} < \underbrace{H_{n+1}\left(\frac{1}{n+1}, ..., \frac{1}{n+1}\right)}_{n+1} \tag{9}$$

4) **Additivity** The amount of entropy should be independent of how the process is regarded as being divided into parts

In a sense Shannon entropy is the simpler way ($K = 1$) to associate a value to the information content of a stochastic variable. See [7] for a broader discussion.

If we had for example hysteresis interactions among the paths (for example in a quantum system) or any kind of history dependent behaviors the form of the relation $S(W)$, S for thermodinamic entropy, and thus of $H(W)$, H for Shannon Entropy, would change, see [5].

In general, the W quantities may require advanced methods from differential geometry and complex analysis in order to be calculated as we will see in the following section IV. However, in some specific cases it can be straightforward as in the example of the harmonic oscillator, which we discuss in the following subsection III-A with the purpose of giving an intuitive understanding of equation 1 to the reader.

A. Example: The Linear Harmonic oscillator

A simple embodied agent is given by the oscillator in Figure 1, a simple linear mass-spring system, where a mass m is attached to a spring with stiffness k. Its motion is governed by the second-order differential equation:

$$m\ddot{x}(t) + kx(t) = 0 \tag{10}$$

Where $x(t)$ is the displacement of the mass as a function of time, $\ddot{x}(t)$ is the acceleration. Its phase portrait can be easily calculated, as shown in III-B. See Figure 2(a).

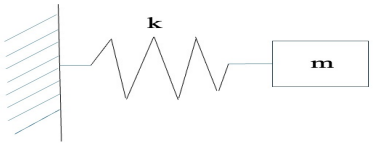


Fig. 1: The schematic representation of a Harmonic Oscillator

Assume that we want the mass to follow a specific desired trajectory $x_d(t)$, which we know in advance. In *feed-forward control* also known as *open loop control* , by using the model, we can calculate the exact force $F_{ff}(t)$ required to move the mass in a way that such that the actual position $x(t)$ matches the desired position $x_d(t)$. The control force will be:

$$F_{ff}(t) = m\ddot{x}_d(t) + kx_d(t) \tag{11}$$

This control action compensates for the natural dynamics of the system and drives the mass along the desired trajectory. When noise is introduced, it affects the motion of the mass-spring system, causing deviations from the expected trajectory. We can model noise in the system as an external disturbance force $N(t)$ added to the dynamics of the system. The equation of motion for the system now becomes:

$$F_{ff}(t) = m\ddot{x}_d(t) + kx_d(t) - N(t) \tag{12}$$

The system will in general, obeying 12, follow a trajectory that can be approximated by $x_d^n(t) \sim x_d(t) + \mathcal{N}(\mu, \sigma^2)$, where $\mu = 0$ and σ depends on the characteristics of the external noise. The phase space representation is qualitatively depicted in 2(b) in section III-B by the filled elliptic annulus.

With noise $N(t)$ in the system, applying $F_{ff}(t)$ as calculated in 11 can lead to control failure and it is in general suboptimal.

To handle the noise we augment the system with *feedback control* also known as emphclosed loop control . The feedback controller reacts to the error caused by the noise and adjust the control force accordingly.

Let $e(t) = x_d(t) - x(t)$ represent the error between the desired trajectory $x_d(t)$ and the actual trajectory x(t) . The feedback controller adjusts the control force based on this error, such as in a proportional-derivative (PD) controller:

$$F_{fb}(t) = K_p e(t) + K_d \dot{e}(t) \tag{13}$$

Where $F_{fb}(t)$ is the feedback control force that compensates for the errors caused by noise, K_p is the proportional gain, which adjusts the force based on the position error, K_d is the derivative gain, which adjusts the force based on the velocity error.

Thus, the total control force applied to the system becomes:

$$F(t) = m\ddot{x}_d(t) + kx_d(t) + K_p(x_d(t) - x(t)) + K_d(\dot{x}_d(t) - \dot{x}(t)) \tag{14}$$

Substituting $F(t)$ into the system equation and accounting for the noise N(t) :

$$m\ddot{x}(t) + kx(t) = m\ddot{x}_d(t) + kx_d(t) + K_p(x_d(t) - x(t)) + K_d(\dot{x}_d(t) - \dot{x}(t)) + N(t) \quad (15)$$

We can assume that a 'well designed' controller (the optimal one) will in general lead to smaller errors than a comparable 'well designed' feed-forward controller. The resulting phase space is qualitatively depicted in Figure 2(c) in section III-B

B. W quantities for the Harmonic Oscillator

The oscillator shown in Figure 1 is an example of a rudimentary embodied agent.

As in [4], by applying equation 16 that denotes energy conservation, we get, in the case of $F(t) = 0$ in equation :

$$E_{tot} = \frac{m\dot{x}^2}{2} + \frac{kx^2}{2} \quad (16)$$

The system traces a closed curve in phase space. The curve's shape is contingent upon m, k, and the initial values of x and its first derivative. Assuming a uniform distribution [0,X] for x and [0,XP] for the initial condition, the phase space volume of equation 1 is determined by the difference in the areas of the ellipses:

$$E_{\text{tot}}^{\max} = \frac{mXP^2}{2} + \frac{kX^2}{2} \quad (17)$$

The equation of the ellipses is:

$$\frac{m\dot{x}^2}{2E_{\text{tot}}^{\max}} + \frac{kx^2}{2E_{\text{tot}}^{\max}} = 1 \quad (18)$$

From which we obtain the semi-axes a and b :

$$a = \sqrt{\frac{2E_{tot}^{\max}}{m}} \qquad b = \sqrt{\frac{2E_{tot}^{\max}}{k}} \quad (19)$$

We finally determine the phase space volume as follows:

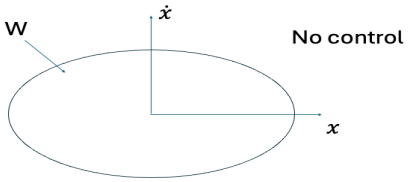
$$W = \pi ab = \pi\sqrt{\frac{2E_{tot}^{\max}}{m}}\sqrt{\frac{2E_{tot}^{\max}}{k}} = \frac{2\pi}{\sqrt{km}}E_{tot}^{\max} \quad (20)$$

Assuming the closed-loop phase region is contained within the open-loop region, the gain K approaches zero in absolute value as these two areas converge, but it tends towards infinity in absolute value while attempting to constrain the system within a diminishing phase space volume (area in this context). In Figure 2 we qualitatively represent the phase spaces of the system in different operating conditions.

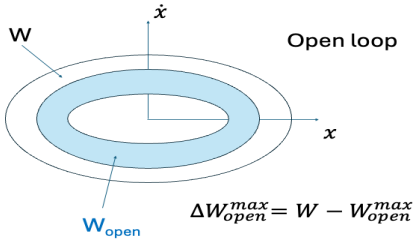
IV. Calculation of W quantities

A. How to calculate the space state volume of the solutions of a Stochastic Partial Differential Equation (SPDE)

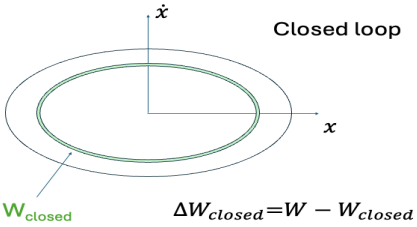
In general, calculating the different quantities involved in equation 1 is technically less direct than in the example above. In this section we discuss the main methods that are available for this task.



(a) Qualitative representation of the phase space of the system in free oscillation when it is not subjected to external forces.



(b) Qualitative representation of the phase spaces for the system subjected to feed-forward (open loop) control.



(c) Qualitative representation of the phase spaces for the system subjected to feedback (closed loop) control

Fig. 2: Phase space representation of the harmonic oscillator for different control conditions. Notice that these pictures are qualitative as due to the mechanical work exerted by the disturbances and the forces actuated by the controllers, the external ellipsis are of similar, but in general different sizes.

Understand the SPDE: analyze the variables, the stochastic components (like noise terms), boundary and initial conditions, and any other relevant parameters.

Numerical Simulation: analytical solutions are often impossible to obtain. You can use numerical methods like finite difference, finite elements, or spectral methods to discretize the SPDEs and simulate them over a range of parameters.

Monte Carlo Methods: Monte Carlo simulations can be particularly useful. By running a large number of simulations with different realizations of the stochastic components, you can obtain a distribution of the solution space.

Estimate State Space Volume: once you have a distribution of solutions from the Monte Carlo simulations, you can start estimating the volume of the state space by statistical techniques (e.g principal component analysis).

High–Dimensional Data Analysis: since the state space is likely high-dimensional, techniques from high-

dimensional data analysis might be necessary. Tools like manifold learning can help in understanding the structure of the solution space.

The interested reader can refer to [8] for more details.

B. Usage of Reproducing Kernel Hilbert Spaces

Reproducing Kernel Hilbert Spaces (RKHS) are an important mathematical concept and are widely used in machine learning, functional analysis, and quantum mechanics. They play a critical role in kernel methods, where they provide a way to map data into a higher-dimensional space where the data is more easily separated or manipulated. Differential equations typically define a flow in phase space. Evaluating the volume of a phase space usually involves integrating over this space. The RKHS approach provides a weighted Bergman kernel function K_γ that maps from $\Omega \times \Omega$ [where Ω is the phase space] to the complex numbers, corresponding to an admissible weight $\gamma \in AW(\Omega)$. One associates to K_γ a measure alternative to Liouville measure, allowing for calculations of volumes relying on complex analysis and complex geometry techniques. Renouncing real valued functions in favor of complex valued functions (of one or several complex variables) is coherent with the philosophy in [9] and [10]. For more details see [11] and [12].

Historical Note. Reproducing kernels were first introduced by the famous Polish mathematician S. Zaremba, and used by him in his work (cf. [13]) on boundary value problems for harmonic and biharmonic functions. The basis for a firm foundation of the theory of RKHS were laid by N. Aronszajn (cf. [14]) and the Moore-Aronszajn theorem remains a cornerstone e.g. in nonlinear signal processing (cf. [15]). Applications of reproducing kernels to signal theory (cf. [16]) go back as far as the work by K. Yao (cf. [17]): results in sampling expansions, minimum energy and non-uniform interpolation, and truncation error bounds, find a unified and natural terrain within RKHS theory.

C. Example

The passive walkers are not soft, however they exploit morphological computation, and they have power needs comparable to humans, [18], [19]. Why their control programs are so simple? Equation (1) allows to answer and to develop a working modeling framework for a new breed of humanoids, like the venerable Ecce in Fig. 3.

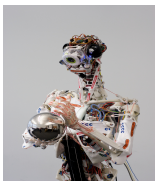


Fig. 3: The 'Ecce' robot.

Calculating the phase space volume of an inverted pendulum involves considering the variables that describe its state and the constraints those variables are subject to.

State Variables For an inverted pendulum, typical state variables include: θ: Angle of the pendulum from the vertical (in radians); $\dot{\theta}$: Angular velocity

Phase Space The phase space is spanned by the above listed variables. So, the phase space point $(\theta, \dot{\theta})$ describes the state of the pendulum.

Constraints The inverted pendulum is subject to physical constraints such as conservation of energy (which determines the total energy of the pendulum) and the dynamics described by its equations of motion (which relate θ and $\dot{\theta}$).

Phase Space Volume To calculate the phase space volume, one needs to integrate over the accessible region in the $(\theta, \dot{\theta})$ space that satisfies the energy constraint and any other relevant constraints. The volume element in phase space is $d\,\mathrm{vol}(g_B)$ where $g_B = \mathcal{K}^* g_{\mathrm{Fubini-Study}}$ and $\mathcal{K} : \Omega \to \mathbb{CP}\big[L^2H(\Omega\,,\,\gamma)\big]$ is a coherent state map [an anti-holomorphic embedding].

Energy Constraint The total energy E of the pendulum is typically given by $E = \frac{1}{2}I\dot{\theta}^2 + mgh\cos(\theta)$, where I is the moment of inertia, m is the mass, g is the acceleration due to gravity, and h is the height of the pendulum's center of mass.
The accessible region $D = \big\{(\theta\,,\,\dot{\theta}) \in \big[0,\,\frac{\pi}{2}\big] \times \mathbb{R} \;:\; 0 \le \dot{\theta} \le \sqrt{\frac{4g}{h}}\,\sin\frac{\theta}{2}\big\}$ in phase space is determined by the energy constraint $E = \text{constant}$.

Calculating the Volume The phase space volume V for a given energy E is then obtained by integrating over the allowed region in $(\theta, \dot{\theta})$ space:

$$V(E) = \int_D d\,\mathrm{vol}(g_B).$$

In summary, calculating the phase space volume of an inverted pendulum involves setting up the energy constraint, determining the allowed region in $(\theta, \dot{\theta})$ space, and then integrating over that region to find the volume.

D. *Classical states* versus *coherent states for the inverted pendulum*

A *weight* is a measurable map $\gamma : D \to [0, +\infty)$. Let $L^2(D,\,\gamma)$ be the Hilbert space of all square integrable functions $f : D \to \mathbb{C}$

$$\int_D |f(z)|^2\,\gamma(z)\,d\mu(z) < \infty$$

with the inner product

$$(f,\,g)_\gamma = \int_D f(z)\,\overline{g(z)}\,d\,\mu(z),$$

where μ is the Lebesgue measure on $\mathbb{R}^2$. Let $L^2H(D,\,\gamma)$ consist of all $f \in L^2(D,\,\gamma)$ that are holomorphic in D. A weight $\gamma \in W(D)$ is *admissible* if i) the evaluation functional $\delta_z : L^2H(D,\gamma) \to \mathbb{C}$, $\delta_z(f) = f(z)$, is continuous, and ii) $L^2H(D,\,\gamma)$ is a closed subspace of $L^2(D,\,\gamma)$. Any closed subspace of a Hilbert space is a Hilbert space itself. The requirements of admissibility (i)-(ii) allow for applying the celebrated Riesz-Fischer theorem so that δ_z may be uniquely represented by an element of $L^2H(D,\,\gamma)$. Let $AW(D)$ be the set of all admissible weights. The set $W(D)$ of all weights is an infinite dimensional Banach manifold modeled on $L^\infty(D\,,\,\mathbb{R})$ and $AW(D) \subset W(D)$ is an open subset. Let $K_\gamma(\zeta\,,\,z)$ be the weighted Bergman kernel of weight γ (the γ-Bergman kernel) associated to an arbitrary

admissible weight $\gamma \in AW(D)$ (cf. [20]). $L^2H(D,\,\gamma)$ is a RKHS with the reproducing kernel $K_\gamma(\zeta\,,\,z)$. The coherent state map $\mathcal{K} : D \to \mathbb{CP}\big[L^2H(D,\,\gamma)\big]$ (associating to every classical state $z \in D$ of the inverted pendulum the coherent state $\mathcal{K}(z)$) is $\mathcal{K}(z) = \big[K_\gamma(\,\cdot\,,\,z)\big]$. Cf. [21]. Here $\mathbb{CP}(\mathcal{H})$ is the complex projective Hilbert space, associated to the (complex) Hilbert space $\mathcal{H}$, and $[\Psi]$ is the ray through $\Psi \in \mathcal{H} \setminus \{0\}$. If $\Psi\,,\,\Phi \in \mathcal{H} \setminus \{0\}$ we also set $\big\langle [\Psi],\,[\Phi]\big\rangle = \Big(\frac{\Psi}{\|\Psi\|}\,,\,\frac{\Phi}{\|\Phi\|}\Big)_{\mathcal{H}}$. Given the classical states $z,\,\zeta \in D$, thought of as identified with the coherent states $\mathcal{K}(z)$ and $\mathcal{K}(\zeta)$, the transition probability amplitude from z to ζ is

$$a_{0\overline{0}}(\zeta\,,\,z) = \big\langle \mathcal{K}(z),\,\mathcal{K}(\zeta)\big\rangle\,.$$

Let $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit ball in the complex plane and let $R : \mathbb{B} \to D$ be the Riemann mapping [the conformal mapping $R : \mathbb{B} \to D$ normalized by $R(0) = z_0$ and $R'(0) > 0$, for a given point $z_0 \in D$]. Let $s = \sigma + it$, $\sigma > -1$, $t \in \mathbb{R}$, and $\varphi_s(\zeta) = \big(1 - |\zeta|^2\big)^s$, $\zeta \in \mathbb{B}$. Then $\varphi_s \in AW(\mathbb{B})$, so that $\gamma_s := \varphi_s \circ R^{-1} \in AW(D)$ and

$$K_{\gamma_s}(\zeta\,,\,z) = \frac{\partial R}{\partial \zeta}(\zeta)\,\overline{\frac{\partial R}{\partial \zeta}(z)}\,K_{\varphi_s}\big(R(\zeta),\,R(z)\big),$$

where (cf. [22])

$$K_{\varphi_s}(\zeta\,,\,z) = \frac{\Gamma(s+2)}{\Gamma(s+1)}\,\big(1 - \zeta\,\overline{z}\big)^{-(s+2)}\,.$$

Let $E = D \times \mathbb{C}$ be the trivial holomorphic line bundle, with the Hermitian metric $H_s(\sigma_0\,,\,\sigma_0) = \gamma_s$, where σ_0 is the section in E given by $\sigma_0(z) = (z,\,1)$. Geometric quantization theory ordinarily (cf. e.g. [23]) works with the Hilbert space $\mathcal{M}$ of all square integrable holomorphic sections in E, rather than $L^2H(D,\,\gamma_s)$, with respect to the Liouville measure

$$d\mu_L = -i\,\omega_{0\overline{0}}\,d\zeta \wedge d\overline{\zeta}\,,$$

$$\omega_{0\overline{0}} = \omega\Big(\frac{\partial}{\partial \zeta}\,,\,\frac{\partial}{\partial \overline{\zeta}}\Big),\ \ \omega = \mathrm{curv}(E,\,\nabla),$$

where ∇ is the canonical Hermitian connection of the Hermitian vector bundle $(E,\,H_s)$. ω is nondegenerate i.e. $(D,\,\omega)$ is a symplectic manifold. Nondegeneracy of ω qualifies $(E,\,H_s)$ as a *quantum bundle*, and quantum bundles are the basic data allowing for the quantization of both the classical states (of the inverted pendulum as a mechanical system whose phase space is the complex manifold D), achieved through the construction of a coherent state map $\mathcal{K} : D \to \mathbb{CP}(\mathcal{M})$, and the quantization of the (quantizable) observables $f \in C^\infty(D)$, a process assigning to each such f a self-adjoint operator $\hat{f} : \mathcal{M} \to \mathcal{M}$ (cf. [24]). The transition probability amplitude from z to ζ with simultaneous transition through w is $a_{0\overline{0}}(w,\,z)\,a_{0\overline{0}}(\zeta,\,w)$ and our construction of the coherent state map $\mathcal{K}$ is "legitimate" if the transition probability amplitude $a_{0\overline{0}}(\zeta,\,z)$ is the average of $a_{0\overline{0}}(w,\,z)\,a_{0\overline{0}}(\zeta,\,w)$ over $w \in D$, with respect to the Liouville measure. The two approaches (in [21] and [23]) can be made to merge by a clever choice of weights of integration i.e. such that

$$\hat{H}_s = C\,H_s \tag{21}$$

for some $C > 0$ where

$$\hat{H}_s\big(\sigma_0\,,\,\sigma_0\big)_z = -\frac{2\,\omega_{0\overline{0}}(z)}{K_{\gamma_s(z,\,z)}}\,,\ \ z \in D.$$

Equation (21) is equivalent to $\gamma = \gamma_s$ obeying to

$$\frac{\partial^2 \log \gamma}{\partial z\,\partial \overline{z}} = 2\pi C\,K_\gamma(z,\,z)\,\gamma(z)$$

(in higher dimensions, a nonlinear complex Monge-Ampére equation). Such a choice $C = C(s) > 0$ exists and

$$a_{0\overline{0}}(\zeta\,,\,z) = i\int_D a_{0\overline{0}}(w,\,z)\,a_{0\overline{0}}(\zeta\,,\,w)\,d\mu_L(w)$$

as required. Again by (21)

$$\mathrm{vol}_L(D) = \int_D d\mu_L(\zeta) =$$

$$= \frac{C(s)}{2}\int_D K_{\gamma_s}(\zeta\,,\,\zeta)\,\gamma_s(\zeta)\,d\zeta \wedge d\overline{\zeta}$$

and the calculation of the area of D is reduced to the calculation of the γ_s-Bergman kernel. It was our main purpose in this section to bring the weighted Bergman kernel into the picture. To be entirely fair to the reader, γ_s-Bergman kernels are rather difficult to compute in close form, yet may be approximated to the desired accuracy, by an approximation of the Riemann mapping. Let us set $D_\epsilon = \{z \in D : \mathrm{dist}(z,\,\partial D) > \epsilon\}$. Let $N \in \mathbb{Z}$, $N \geq 1$, and let T_N be the union of a minimal collection of squares of side $\geq 2^{-N}$ that lie in D, and the vertices of the squares in T_N that belong to ∂T_N form a polygon Γ_N such that its interior P_N satisfies $P_N \supset D_{1/N}$. If $R_N : \mathbb{B} \to P_N$ is the Riemann mapping of D_N then

$$R_N'(z) = \frac{R_N'(0)}{\prod_{j=1}^N \big(1 - \overline{z}_j\,z\big)^{\alpha_j - 1}}\,,$$

where w_j are the vertices of Γ_N, each of interior angle $\pi\alpha_j$, and $w_j = R_N(z_j)$. Cf. [25], p. 236. Then $D_N \to D$ as $N \to \infty$ (in the sense of Carathéodory, [26]) so that (by a result in [27]) $R_N \to R$ as $N \to \infty$, uniformly on the compact subsets of $\mathbb{B}$. Let $\gamma_{s,N} = \varphi_s \circ R_N^{-1}$. Then (by the weighted version of Ramadanov's theorem, cf. [28]) $\lim_{N\to\infty} K_{\gamma_{s,\,N}} = K_{\gamma_s}$ locally uniformly on $D \times D$. Fast algorithms for computing the Schwarz-Christoffel maps R_N are also available (cf. [29]).

Historical Note. Bergman kernels (unweighted i.e. $\gamma \equiv 1$) are due to Steven Bergman, [30], and an early account of their wide applicability is furnished in the monograph [31]. In 1952 Gaetano Fichera started (cf. [32]) a personal crusade against Bergman kernels, on the ground of their being "inessential" *vis-a-vis* to other methods in Mathematical Analysis. Bergman kernels fit into the general theory of RKHS, which was already available (cf. [14]) at the time Fichera's paper [32] was written. The further growth of the use of kernel methods in the theory of PDEs (cf. e.g. [33], [34]), complex geometry (relative to Bergman metrics, cf. [35], pp. 367-371) and approximation theory (cf. e.g. [36], [37]) suggest that the great Italian scientist was for once abandoned by his undiscussable mathematical intuition.

V. Remarks

Equation (1) relies on very general assumptions and has a wide applicability. It provides a useful method to link the system dynamics of a robot to the computational effort needed to control it. It can explain, model, and measure the benefits - and shortcomings - of 'softness' in a given robot (or parts of it) for a given application, by following the approach proposed in [38] . It allows to do so by looking at how the morphology shapes the phase space and consequently the information processing and the intelligent behaviors of a physical agent. However, the approach to calculate the phase space volumes for a given soft physical embodiment (that in general will lead to a more or less complicated set of Stochastic Partial Differential Equations) may not seem straightforward. We are developing novel approaches based on contemporary Differential Geometry and Complex Calculus allowing to do that and we are integrating the dynamics of the agents in a systematic way into physics and information informed learning systems for the control of soft embodied intelligent agents.

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